adequate pump NPSH, the ability to predict maximum temperatures encountered at the pump inlet is of considerable value.

References

[1] Eckert, E. R. G. and Jackson, T. W., "Analysis of turbulent free convection boundary layer on a flat plate," NACA Rept. 1015 (July 12, 1950).

[2] McAdams, W. H., *Heat Transmission* (McGraw-Hill Book Co. Inc., New York, 1954), 3rd ed., p. 172.

Validity of Series Expansions of Kepler's Equation

ERWIN W. PAUL*

Westinghouse Electric Corporation, Baltimore, Md.

A solution of the Kepler equation $M = E - e \sin E$ is discussed. In digital computation, very often preference is given to a Taylor expansion progressing in powers of the time. The complex representation of the Kepler equation as an analytical function shows clearly the limited interval of convergence.

THE complete solution of the problem of the motion of a satellite in its elliptical orbit includes the solution of the Kepler equation

$$M = E - e \sin E$$

The mean anomaly, M, is essentially a time measure, and E and e are the eccentric anomaly and the eccentricity, respectively.

The Kepler equation is transcendental in E. Therefore, an analytical presentation of $E = E(M,e)$ can be found only approximately by some series expansion of E. Two expansions are common, namely:

1) The Fourier-Bessel expansion, progressing in multiples of the variable M:

$$E = M + 2\sum_{p=1}^{\infty} \frac{Jp(pe)}{p} \sin(pM)$$

The Bessel coefficients depend on the fixed value of e. This expansion is valid for $0 \leq e < 1$ for all values of M.

2) The Taylor expansion of E, progressing in powers of the variable e:

$$E = \sum_{p=0}^{\infty} B_p(M)e^p$$

The coefficients depend on the fixed value of M. This expansion is due to Lagrange and converges for $0 \leq e < 0.6627$ for all values of M. The details of such expansions can be found, for example, in Refs. 1 and 2.

In digital computation, very often preference is given to Taylor expansions progressing in powers of the time, or equivalently, in powers of the mean anomaly. Of course, the validity of such an expansion is likewise restricted to the region of convergence which can best be found by means of the theory of analytical functions. It is important to know the radius of convergence, which is equal to the distance from the point about which the expansion is performed to the nearest singularity, which in the complex representation is mapped as a critical point. In the following, the convergence of this type of Taylor expansion of E is discussed, namely:

3) The Taylor expansion of E, progressing in powers of M:

$$E = \sum_{p=0}^{\infty} C_p(e)(M - M_1)^p$$

The coefficients depend on the fixed value of e. In order to simplify the expression for the mean anomaly, M, the convention is made that the satellite passes perigee at time zero. Then the mean anomaly is $M = 2\pi(t/T)$, where T is the period of revolution and t is the time.

Introducing the time ratio $\tau = t/T$ rather than the mean anomly M, the Kepler equation is written as

$$\tau = (1/2\pi)(E - e \sin E) \quad (1)$$

and the series expansion of E becomes

$$E = \sum_{p=0}^{\infty} D_p(e)(\tau - \tau_1)^p \quad (2)$$

Let $E = x + iy$ and $\tau = u + iv$ be the points of the E and τ complex planes, respectively. Equation (1) then is considered as an analytic function that maps the E plane on the τ plane conformally (angle preserving). The conformality is interrupted at the singular points of the function. Denoting

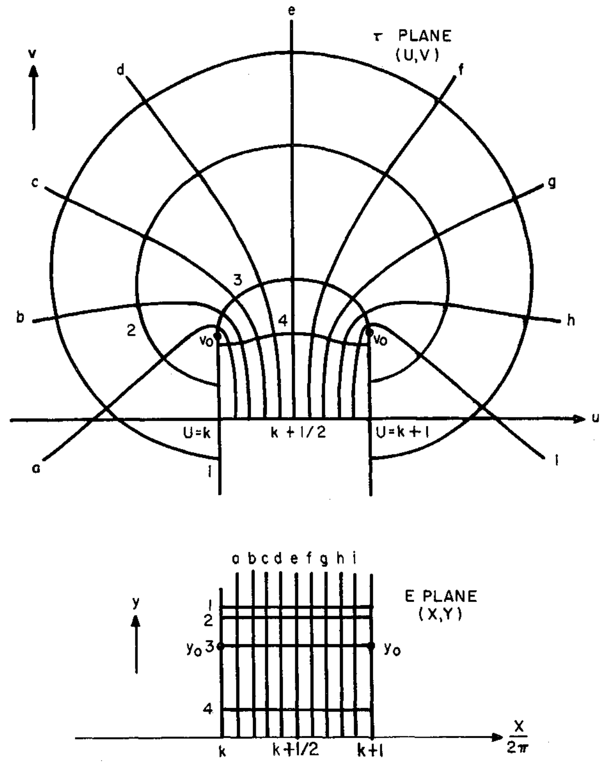

Fig. 1 The conformal mapping of x = const and y = const

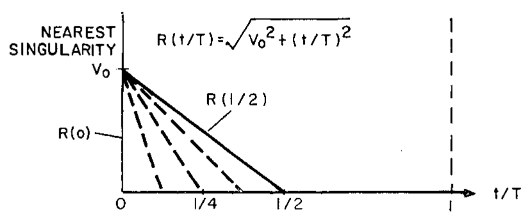

Fig. 2 The radius of convergence of one value of e

Received by ARS October 23, 1962; revision received March 29, 1963. This research was sponsored by the U. S. Air Force under Contract AF 49(638)1002 monitored by the Office of Scientific Research of the Office of Aerospace Research. The writer is indebted to George Shapiro, Advisory Mathematician of the Westinghouse Air Arm Division, for his helpful suggestions and criticisms during the preparation of this paper. He also thanks the reviewer for calling Ref. 3 to his attention.

* Fellow Engineer, Computing and Control Systems, Air Arm Division.

the singularities in either plane by the subscript zero, one obtains from Eq. (1)

$$x_0 = 2\pi k \qquad k = 0, \pm 1, \pm 2, \ldots$$

$$y_0 = \cosh^{-1}\left(\frac{1}{e}\right) = \pm \ln\left[\frac{1 + (1 - e^2)^{1/2}}{e}\right]$$

as singular points in the E plane and

$$u_0 = k \qquad k = 0, \pm 1, \ldots$$

$$v_0 = \pm \frac{1}{2\pi}\left\{\ln\left[\frac{1 + (1 - e^2)^{1/2}}{e}\right] - (1 - e^2)^{1/2}\right\} \tag{3}$$

as critical points in the τ plane.

Clearly the singular points of the E plane are those points the real parts of which are integer multiples of 2π, lying on the two horizontal lines at a distance y_0 above and below the real (x) axis. The critical points of the τ plane are located similarly at a distance v_0 above and below the real (u) axis. In addition to this doubly infinite set, the point at infinity of each plane is a nonisolated singularity.

Thus the region of regularity in the E plane is a strip parallel to the real axis and bounded by two lines formed by two sets of singular points. The height, $2y_0$, of this strip depends on the value of e. For $e = 1$ it is zero, and no region of regularity exists. For $e = 0$ the height is infinite. For $0 < e < 1$, a limited region of regularity exists. The corresponding region of regularity in the τ plane is bounded by the images of the horizontal lines $y_0 =$ const. These images are arcs going through the critical points v_0, as can be seen from Fig. 1. It may be mentioned that uniqueness is achieved by representation on the Riemann surface, e.g., Refs. 2 and 3.

Now, an analytic function always can be expanded in a Taylor series in its region of regularity. Let E be expanded in a Taylor series about a real time point τ_1, as in (2). This expansion will converge only in the region that is inside the circle of convergence about the chosen point t_1/T. The radius of this circle is the distance between the real time point t_1/T on the u axis, about which the expansion is made, and the singularity nearest to this point. Any real time point between perigee and apogee might be taken. The distance, v_0, of the singularities from the real time axis was given in (3) as a function of e. Therefore, the circle of convergence depends on the eccentricity (e) as well as on the chosen time point (t_1) about which the eccentric anomaly (E) is expanded.

Figure 2 shows the radius of convergence $R(t_1/T)$, which is greatest for an expansion at apogee ($t_1/T = \frac{1}{2}$) and smallest for an expansion at perigee ($t_1/T = 0$). Thus $v_0 \leqslant R \leqslant (v_0^2 + \frac{1}{4})^{1/2}$. This limitation of the radius of convergence, $R(t_1/T)$, covers all expansions about any real time point, t_1, during one period, T, of revolution. Figure 3 shows the radius of convergence as a function of the eccentricity, e, applied to expansions at perigee and at apogee. At other points the radius of convergence is between the values shown in the figure.

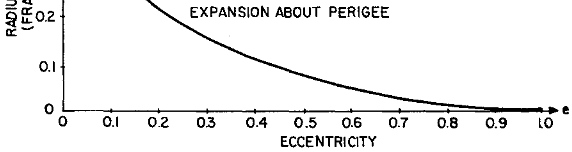

Fig. 3 Limiting values of radius of convergence

These considerations for the two-body problem emphasize the importance of investigating the analytic character of a function, in particular its region of regularity. The forementioned results indicate that a Taylor expansion must be used cautiously, especially about points near perigee of an elliptical orbit with great eccentricity. For example, when $e = 0.9$, the expansion at perigee converges only for a time interval $|\Delta t| = 0.005\ T$. Such a case occurs at injection into an earth-moon orbit, although this becomes a three-body problem.

References

[1] Moulton, F. R., *An Introduction to Celestial Mechanics* (The Macmillan Co., New York, 1959), Chap. V, pp. 165–169.

[2] Wintner, A., *Analytical Foundations of Celestial Mechanics* (Princeton University Press, Princeton, N. J., 1947), Chap. IV, pp. 212–222.

[3] Cherry, T. M., "On Kepler's equation," Proc. Cambridge Phil. Soc. 51, 81–91 (1955).

Graphical Evaluation Of the Trade-Off between Specific Impulse and Density

Morton A. Klotz*
Aerojet-General Corporation, Azusa, Calif.

GORDON[1] has shown that the relative effects of propellant specific impulse I_s and density in a volume-limited vehicle may be evaluated by the equation

$$\Omega = I_s(\rho/\rho_0)^K \tag{1}$$

where ρ is the density of the propellant, ρ_0 the density of a reference propellant, K a fraction between 0 and 1.0 calculated from the design parameters of a given vehicle, and Ω is defined as the "effective specific impulse."

The parameter K is not a constant; for a given stage it depends on the weight of the higher stages and therefore on the density of the propellant in the higher stages. Exact values of K are difficult to calculate, and it is better to calculate burnout velocities and ranges if precise results are needed for a given vehicle. Nevertheless, Eq. (1) is useful for preliminary propellant evaluation in propellant development programs. For this purpose it is satisfactory to use the K values given by Gordon for an ideally staged volume-limited vehicle with propellant mass fractions of 0.9 in each stage and a stage ratio of 3 (see Table 1).

The usefulness of Eq. (1) is enhanced by plotting Ω semilogarithmically against K, as in Fig. 1. Since

$$\log\Omega = K \log(\rho/\rho_0) + \log I_s \tag{2}$$

such a plot yields a straight line of slope $\log(\rho/\rho_0)$ and intercept I_s. Values of Ω for any value of K between 0 and 1.0 thus are obtained by drawing a straight line between the I_s on the left ($K = 0$) and $I_s\ (\rho/\rho_0)$ on the right ($K = 1.0$). In such a plot, a propellant with $\rho = \rho_0$ appears as a horizontal

Received by IAS December 3, 1962; presented at the AIAA Summer Meeting, Los Angeles, Calif., June 17–20, 1963. Information contained herein was developed for the U. S. Government. The U. S. Government may publish such information at any time. A similar plot for the related equation $\Omega = I_s\rho^K$ was used in an earlier, unpublished work by L. J. Gordon, to whom the author is grateful for helpful advice and suggestions.

* Assistant Senior Chemist, Solid Propellant Research Operations.